JOURNAL OF APPROXIMATION THEORY **27**, 329–345 (1979)

Limit-Periodic General T-Fractions and Holomorphic Functions

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Communicated by Oved Shisha

Received May 15, 1978

1. INTRODUCTION

We shall use the symbol K to denote a continued fraction (terminating or non-terminating),

$$\mathop{\mathrm{K}}_{n=1}^{N} \frac{a_n}{b_n} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_N}{b_N} = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_N}{b_N}}},$$

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n}{b_n} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}}.$$

The general T-fraction

$$\mathop{\mathrm{K}}_{n=1}^{N} \frac{F_n z}{1 + G_n z}, \qquad F_n \neq 0 \text{ for } n < N+1, \qquad N \leqslant \infty, \tag{1.1}$$

[5, p. 173], [2], [4], is said to *correspond to* the pair $(L(z), L^*(z))$ of formal Laurent series

$$\begin{aligned} L(z) &= c_1 z + c_2 z^2 + c_3 z^3 + \cdots \\ L^*(z) &= c_0^* + c_{-1}^* z^{-1} + c_{-2}^* z^{-2} + \cdots \end{aligned} \tag{1.2}$$

if and only if for any natural number n the nth approximant of (1.1) has a

* This research was supported by the Norwegian Research Council for Science and the Humanities.

0021-9045/79/120329–17$02.00/0

Maclaurin expansion that agrees with $L(z)$ up to and including the term $c_n z^n$, and a Laurent expansion at ∞ that agrees with $L^*(z)$ down to and including the term $c^*_{-n+1} z^{-n+1}$. (For finite N all nth approximants for $n \geqslant N$ are equal to the Nth approximant.)

In the paper [2] there is a set of necessary and sufficient conditions on the coefficients of $L(z)$ and $L^*(z)$ for existence of a corresponding non-terminating (i.e., $N = \infty$) general T-fraction with all $G_n \neq 0$. It is rather easy to prove that $G_n \neq 0$ for $n < N + 1$ is necessary (and sufficient) for a general T-fraction (1.1) to correspond to some pair (1.2) of formal Laurent series, and hence the condition $G_n \neq 0$ does not represent any restriction.

In the paper [7] it is proved that if $L(z)$ and $L^*(z)$ actually represent *functions* in sufficiently large neighborhoods of 0 and ∞ and satisfy certain boundedness conditions, then a corresponding general T-fraction exists and converges to $L(z)$ and $L^*(z)$ locally uniformly in certain neighborhoods of 0 and ∞. A crucial point in the argument is that the boundedness conditions imply that F_k and $-G_k$ both tend to a limit $F \neq 0$ as $k \to \infty$ in the following way: For a certain $C > 0$ and $\theta \in (0, 1)$ we have for all k

$$| F_k - F | \leqslant C \cdot \theta^k, \qquad | G_k + F | \leqslant C \cdot \theta^k. \tag{1.3}$$

This result shows that it is natural to study general T-fractions (1.1) with the property (1.3) and ask for properties of $L(z)$ and $L^*(z)$.

The purpose of the present paper is to carry out such an investigation. Correspondence will be the main issue, but convergence will also be touched upon.

2. Notations. Definition. Statement of the Main Result

Some formulas will take a more convenient form if we in $L(z)$ replace c_n by $(-1)^{n+1} \gamma_n$ and in $L^*(z)$ replace c_n^* by $(-1)^{n+1} \gamma_n^*$, in which case $L(z)$ and $L^*(z)$ are written

$$\begin{aligned} L(z) &= \sum_{n=1}^{\infty} (-1)^{n+1} \gamma_n z^n \\ L^*(z) &= \sum_{-\infty}^{n=0} (-1)^{n+1} \gamma_n^* z^n. \end{aligned} \tag{2.1}$$

If the general T-fraction (1.1) (with $N \geqslant 2$) corresponds to the pair $(L(z), L^*(z))$ of formal Laurent series (2.1), then the T-fraction

$$\mathop{\mathrm{K}}_{n=2}^{N} \frac{F_n z}{1 + G_n z}$$

corresponds to a pair $(\tilde{L}(z), \tilde{L}^*(z))$ of formal Laurent series

$$\begin{aligned} \tilde{L}(z) &= \sum_{n=1}^{\infty} (-1)^{n+1} \tilde{\gamma}_n z^n \\ \tilde{L}^*(z) &= \sum_{-\infty}^{n=0} (-1)^{n+1} \tilde{\gamma}_n^* z^n \quad [8]. \end{aligned} \tag{2.2}$$

In case of correspondence the γ_n's and γ_n^*'s can be expressed in terms of the F_k's and the G_k's. The formulas for the $\tilde{\gamma}_n$'s and the $\tilde{\gamma}_n^*$'s are obtained from the first ones by increasing the subscripts of all F_k's and G_k's by *one*.

Since we are going to study the general T-fractions where a condition (1.3) holds, we need a notation for the family of such continued fractions. Without loss of generality we may assume that $F = 1$ (else replace Fz by z'). It will furthermore turn out to be convenient to write $1/R$, $R > 1$, instead of θ.

DEFINITION. For fixed $R > 1$ and $C \in (0, R)$ let $\mathscr{T}_{R,C}$ denote the family of all general T-fractions

$$\operatorname*{K}_{n=1}^{\infty} \frac{F_n z}{1 + G_n z}$$

with the property that for all n

$$|F_n - 1| \leqslant \frac{C}{R^n} \quad \text{and} \quad |G_n + 1| \leqslant \frac{C}{R^n}.$$

Remark. The condition $C < R$ implies that all F_n and G_n are $\neq 0$. Hence all general T-fractions in $\mathscr{T}_{R,C}$ are non-terminating and correspond to some pair (2.1) of formal Laurent series [8].

The following theorem is the main result of the paper. The other results are simple consequences of this theorem.

THEOREM 1. *Given $R > 1$ and $\epsilon \in (0, 1 - 1/R)$. Then there is a $C > 0$ such that any general T-fraction in $\mathscr{T}_{R,C}$ corresponds at the origin to the Maclaurin expansion of a function, holomorphic in $|z| < R - \epsilon$, and at ∞ to the Laurent expansion of a function, holomorphic in $|z| > 1/R + \epsilon$ (also at ∞).*

The statement is sharp in the following sense: To any $C \in (0, R)$ there is a general T-fraction in $\mathscr{T}_{R,C}$ whose corresponding series at the origin has a radius of convergence at most $=R$, and whose corresponding series at ∞ diverges for all z with $|z| < 1/R$.

In Section 3 of the paper the statement about correspondence at 0 will be proved, and in Section 4 the statement about correspondence at ∞. The proofs of Lemmas 1 and 4 are based upon ideas from [3]. Section 5 will contain an extension of Theorem 1. Section 6 is a brief discussion of convergence problems.

3. Correspondence at 0

Since all general T-fractions in $\mathscr{T}_{R,C}$ are non-terminating and with all $G_n \neq 0$, we may without loss of generality write

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{F_n z}{1 + G_n z}, \qquad F_n \neq 0, \tag{3.1}$$

and we know from [8] that they all have a corresponding pair (2.1) of formal Laurent series. We know furthermore, that

$$\mathop{\mathrm{K}}_{n=2}^{\infty} \frac{F_n z}{1 + G_n z} \tag{3.2}$$

has a corresponding pair (2.2) of formal Laurent series.

The first lemma does not require the general T-fraction to be in $\mathscr{T}_{R,C}$, but it requires correspondence, which e.g. is there if the general T-fraction is in $\mathscr{T}_{R,C}$.

Lemma 1. *If the general T-fraction* (3.1) *corresponds to the pair* (2.1) *of formal Laurent series, then the following set of formulas hold*:

$$\gamma_1 = F_1, \qquad \gamma_{n+1} = G_1 \gamma_n + \sum_{k=1}^{n} \tilde{\gamma}_{n+1-k} \gamma_k$$

$$= (G_1 + F_2)\, \gamma_n + \sum_{k=1}^{n-1} \tilde{\gamma}_{n+1-k} \gamma_k, \qquad n \geqslant 1. \tag{3.3}$$

Proof. From the formal identity

$$\sum_{k=1}^{\infty} (-1)^{k+1} \gamma_k z^k = \frac{F_1 z}{1 + G_1 z + \sum_{k=1}^{\infty} (-1)^{k+1} \tilde{\gamma}_k z^k}$$

it follows:

$$\left(\sum_{k=1}^{\infty} (-1)^{k+1} \gamma_k z^{k-1} \right) \cdot \left(1 + (G_1 + \tilde{\gamma}_1) z + \sum_{k=2}^{\infty} (-1)^{k+1} \tilde{\gamma}_k z^k \right) = F_1 .$$

Comparing coefficients, we first get

$$\gamma_1 = F_1 \qquad \text{(and hence } \tilde{\gamma}_1 = F_2\text{)}$$

and next

$$(-1)^{n+2}\gamma_{n+1}+(-1)^{n+1}(G_1+\tilde{\gamma}_1)\,\gamma_n+(-1)^{n+1}\sum_{k=1}^{n-1}\tilde{\gamma}_{n+1-k}\gamma_k=0,\quad n\geqslant 1.$$

A slight rearrangement gives the second formula, and hence the lemma is proved.

Since we mainly shall be interested in general T-fractions in $\mathscr{T}_{R,C}$, where the F_k's are close to 1 and the G_k's are close to -1, it will be of advantage to put

$$F_k=1+f_k,\qquad G_k=-1+g_k. \tag{3.4}$$

The formulas for the first γ_k's are then:

$$\begin{aligned}\gamma_1&=1+f_1\\ \gamma_2&=(f_2+g_1)(1+f_1)\\ \gamma_3&=(f_2+g_1)\,\gamma_2+\tilde{\gamma}_2\gamma_1\\ &=(f_2+g_1)^2(1+f_1)+(f_3+g_2)(1+f_2)(1+f_1).\end{aligned}$$

Also in the next lemma we do not require the general T-fraction to be in $\mathscr{T}_{R,C}$, but of course we maintain the correspondence requirement.

LEMMA 2. *If for all $n\geqslant 1$ $f_n>0$ and $g_n>0$, then*

$$\gamma_n\geqslant(1+f_1)\cdots(1+f_{n-1})(f_n+g_{n-1})$$

for all $n\geqslant 2$.

Proof. We first observe that under the conditions of the lemma all γ_n's must be positive. For $n=1,2,3$, this is readily seen from the formulas, and it follows generally by a simple induction argument. (Of course all $\tilde{\gamma}_n$'s must also be positive.)

From the formulas above we see directly that the inequality holds for $n=2$ and $n=3$. Let $N\geqslant 3$ be a number such that the inequality holds for $n=2,3,\ldots,N$ (and of course $\tilde{\gamma}_n\geqslant(1+f_2)\cdots(1+f_n)(f_{n+1}+g_n)$ holds for the same n-values). Then we have, since according to the remark on the positivity of the γ_n's the omitted terms in the last formula (3.3) all must be positive:

$$\gamma_{N+1}>\tilde{\gamma}_N\gamma_1\geqslant(1+f_1)(1+f_2)\cdots(1+f_N)(f_{N+1}+g_N).$$

Hence Lemma 2 is proved by induction.

PROPOSITION 1. *For any $C\in(0,R)$ there is in $\mathscr{T}_{R,C}$ a general T-fraction whose corresponding series at 0 has a radius of convergence at most equal to R.*

Proof. Take the general T-fraction with

$$f_k = g_k = \frac{C}{R^k} \qquad \text{for all } k, \tag{3.5}$$

and let

$$\sum_{n=1}^{\infty} (-1)^{n+1} q_n z^n \tag{3.6}$$

be the corresponding power series at the origin. Then all q_n are positive, and from Lemma 2 it follows that for all $n \geqslant 2$

$$q_n > f_n = \frac{C}{R^n}.$$

This proves that the radius of convergence of (3.6) cannot exceed R, and hence the Proposition 1 is proved.

Proposition 1 takes care of the "sharpness statement" in Theorem 1 as far as correspondence at 0 is concerned.

In order to prove the "holomorphity statement" we need upper bounds for the γ_n's.

In the next lemma we only require correspondence. But for our purpose the lemma will be most useful for $f_k > 0$, $g_k > 0$, in particular in the case $f_k = g_k = C/R^k$.

LEMMA 3. *For any $n \geqslant 2$, γ_n is a sum of products of factors of the types $(1 + f_m)$ and $(f_{k+1} + g_k)$, and every term contains at least one factor of the type $f_{k+1} + g_k$.*

Proof. From the expressions for γ_2 and γ_3 we see immediately that the statement of the lemma holds for $n = 2$ and $n = 3$. It is also obvious that if it holds for some γ_n, then it also holds for $\tilde{\gamma}_n$.

Let $N \geqslant 3$ be a number such that the statement of the lemma holds for $n = 2, 3, \dots, N$. Then it obviously holds for the expression

$$(g_1 + f_2)\gamma_N + \sum_{k=1}^{N-1} \tilde{\gamma}_{N+1-k}\gamma_k,$$

and hence for γ_{N+1} (see (3.3)). The lemma is thus proved by induction.

Three important properties follow from Lemma 3:

(1) If $f_k > 0$ and $g_k > 0$ for all k then all γ_n are positive. (This, however, is something we already know from the proof of Lemma 2.)

(2) If $f_k > 0$ and $g_k > 0$ for all k, then for all n, γ_n is a strictly increasing function of all f_k's and g_k's in the formula for γ_n. On the other hand,

for fixed absolute value of all f_k and g_k the $|\gamma_n|$'s are maximal when $f_k \geqslant 0$ and $g_k \geqslant 0$ for all k. From this follows in particular that in $\mathscr{T}_{R,C}$ we generally have for all n

$$|\gamma_n| = q_n,$$

where the q_n's are defined in (3.6).

(3) For all $n \geqslant 2$ we have

$$\tilde{q}_n \leqslant \frac{q_n}{R}. \tag{3.7}$$

From the properties 1 and 3 it follows by using Lemma 1 that

$$q_{n+1} \leqslant \left(\frac{C}{R} + \frac{C}{R^2}\right) q_n + \frac{1}{R} \sum_{k=1}^{n-1} q_{n+1-k} q_k \tag{3.8}$$

for all $n \geqslant 1$.

Let $\langle t_n \rangle_{n=1}^{\infty}$ be a sequence of positive numbers, defined by

$$\begin{aligned} t_1 &= q_1 = 1 + \frac{C}{R} \\ t_{n+1} &= \left(\frac{C}{R} + \frac{C}{R^2}\right) t_n + \frac{1}{R} \sum_{k=1}^{n-1} t_{n+1-k} t_k, \qquad n \geqslant 1. \end{aligned} \tag{3.9}$$

Then, from property 2 and (3.8) it follows that for all continued fractions in $\mathscr{T}_{R,C}$ we have

$$|\gamma_n| \leqslant t_n \tag{3.10}$$

for all n.

We shall now study the formal power series

$$T(z) = \sum_{n=1}^{\infty} t_n z^n. \tag{3.11}$$

The recursion formula can be rewritten in the following form:

$$t_{n+1} = \frac{C}{R} t_1 t_n + \frac{1}{R} \sum_{k=1}^{n-1} t_{n+1-k} t_k, \qquad n \geqslant 1.$$

From this it follows that $T(z)$ satisfies the following formal identity:

$$T(z) - t_1 z = \frac{C-1}{R} t_1 z T(z) + \frac{1}{R} T(z)^2.$$

We rearrange the identity:

$$T(z)^2 - [R + (1 - C)\, t_1 z]\, T(z) + R t_1 z = 0.$$

From this and $T(0) = 0$ it follows that

$$T(z) = \tfrac{1}{2}[[R + (1 - C)\, t_1 z] - (R^2 - 2R(1 + C)\, t_1 z + (1 - C)^2\, t_1^2 z^2)^{1/2}]. \tag{3.12}$$

This shows that $T(z)$ is not only a formal power series, but that it represents a holomorphic function in a disk centered at the origin. The radius of the disk is equal to the distance from the origin to the nearest singularity z_0, which in this case is a branch point, i.e., the root of the equation

$$(1 - C)^2\, t_1^2 z^2 - 2R(1 + C)\, t_1 z + R^2 = 0$$

with smallest absolute value. Simple calculation yields (also for $C = 1$).

$$z_0 = \frac{R}{(1 + C^{1/2})^2 (1 + C/R)} \cdot \tag{3.13}$$

For given R and ϵ all sufficiently small values of C will make $|z_0| > R - \epsilon$. Letting C have a such value we know that for any continued fraction in $\mathscr{T}_{R,C}$ the corresponding series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\, \gamma_n z^n \tag{3.14}$$

has coefficients dominated by those of $T(z)$ (see (3.10)), and hence, by a trivial argument, the series (3.14) must also represent a holomorphic function in $|z| < R - \epsilon$. Theorem 1 is thus proved as far as correspondence at 0 is concerned. Before going over to correspondence at ∞ we shall make two observations, the first one of use in Section 5 for the extension of Theorem 1, the second one of use in the convergence discussion in Section 6.

Observation 1. For any continued fraction in $\mathscr{T}_{R,C}$ the corresponding series at the origin represents a holomorphic function in some neighborhood of the origin. (At least in $|z| < R/[2(1 + R^{1/2})^2]$, as may be seen from (3.13).)

Observation 2. Let $R > 1$ and $\epsilon \in (0, R - 1)$ be given. Let furthermore C be such that $T(z)$ is holomorphic in $|z| < R - \epsilon$. Let finally $r \in (1, R - \epsilon)$. For any continued fraction in $\mathscr{T}_{R,C}$ the corresponding series $L(z)$ at the origin is holomorphic in $|z| < R - \epsilon$ and satisfies in $|z| \leqslant r$ the following inequality

$$
\begin{aligned}
| L(z) - c_1 z | &= \left| \sum_{k=2}^{\infty} c_k z^k \right| \leqslant \sum_{k=2}^{\infty} t_k r^k = T(r) - t_1 r \\
&= \frac{2 C t_1^2 r^2}{R - (1 + C) t_1 r + (R^2 - 2R(1 + C) t_1 r + (1 - C)^2 t_1^2 r^2)^{1/2}} .
\end{aligned} \tag{3.15}
$$

(It follows from (3.13) and $r < | z_0 |$ that $R - (1 + C) t_1 r > 0$.)

4. Correspondence at ∞

Lemma 4. *If the general T-fraction* (3.1) *corresponds to the pair* (2.1) *of formal Laurent series, then the following set of formulas holds*:

$$
\gamma_0^* = -\frac{F_1}{G_1} = -\frac{\gamma_1}{G_1}, \quad \gamma_{n-1}^* = \frac{-1}{G_1}\left[-\gamma_n^* + \sum_{k=n}^{0} \tilde{\gamma}_{n-k}^* \gamma_k^*\right] \tag{4.1}
$$

$$
= \frac{-1}{G_1}\left[\gamma_n^*(\tilde{\gamma}_0^* - 1) + \sum_{k=n+1}^{0} \tilde{\gamma}_{n-k}^* \gamma_k^*\right], \quad n \leqslant 0.
$$

The proof is a straightforward computation, similar to the one in the proof of Lemma 1, and shall be omitted here.

With f_k and g_k as in (3.4), the formulas for the first γ_k^*'s are:

$$
\gamma_0^* = \frac{1 + f_1}{1 - g_1}
$$

$$
\gamma_{-1}^* = \frac{(1 + f_1)(f_2 + g_2)}{(1 - g_1)^2 (1 - g_2)}
$$

$$
\gamma_{-2}^* = \frac{(1 + f_1)(f_2 + g_2)^2}{(1 - g_1)^3 (1 - g_2)^2} + \frac{(1 + f_1)(1 + f_2)(f_3 + g_3)}{(1 - g_1)^2 (1 - g_2)^2 (1 - g_3)} .
$$

Lemma 5. *If $f_k > 0$ and $0 < g_k < 1$ for all $k \geqslant 1$, then*

$$
\gamma_{-N}^* \geqslant (1 + f_1) \cdots (1 + f_N)(f_{N+1} + g_{N+1})
$$

for all $N \geqslant 1$.

Proof. We first observe that under the conditions of the lemma all γ_n^*'s ($n \leqslant 0$) are positive. For $n = 0, -1, -2$ this is readily seen from the formulas above, and it follows generally by a simple induction argument. (Of course all $\tilde{\gamma}_n^*$'s must also be positive.)

From the formulas above we see directly that the inequality holds for $N = 1$ and $N = 2$. Assume that N is a natural number such that the inequality holds for all γ_{-m}^* with $m \leqslant N$. Since all γ_n^*'s and $(\tilde{\gamma}_0^* - 1)$ are positive and $-1/G_1 > 1$ we have from (4.1)

$$\gamma_{-N-1}^* \geqslant \tilde{\gamma}_{-N}^* \cdot \gamma_0^* > (1 + f_2) \cdots (1 + f_{N+1})(f_{N+2} + g_{N+2})(1 + f_1).$$

Hence the lemma is proved by induction.

Proposition 2. *For any $C \in (0, R)$ there is in $\mathscr{T}_{R,C}$ a general T-fraction whose corresponding series at ∞ diverges for all z in $|z| < 1/R$.*

Proof. Take the general T-fraction with $f_k = g_k = C/R^k$ for all k, and let

$$\sum_{-\infty}^{n=0} (-1)^{n+1} q_n^* z^n \tag{4.2}$$

be the corresponding Laurent series at ∞. All q_n^* are positive, and from Lemma 5 it follows that for all $N \geqslant 1$

$$q_{-N}^* > f_{N+1} = \frac{C}{R^{N+1}}.$$

For any z in $|z| < 1/R$ we have $|q_{-N}^* z^{-N}| > C/R$, and hence (4.2) diverges for such a z, and Proposition 2 is proved (with the same general T-fraction as in Prop. 1).

Proposition 2 takes care of the "sharpness statement" in Theorem 1 for correspondence at ∞.

Lemma 6. *For any $n \leqslant -1$, γ_n^* is a sum of fractions, where the denominator is a product of factors of the form $1 - g_k$ and the numerator is a product of factors of the types $(1 + f_m)$ and $(f_k + g_k)$. Every term contains at least one factor of the type $(f_k + g_k)$.*

Proof. From the formulas we see directly that the statement holds for $n = -1$ and $n = -2$. Let $N \leqslant -2$ be a number such that the statement holds for $n = -1, n = -2, \ldots, n = N$. Then it holds for the expressions

$$-\frac{1}{G_1} \sum_{k=N+1}^{0} \tilde{\gamma}_{n-k}^* \gamma_k^*$$

and

$$-\frac{1}{G_1} \gamma_N^* (\tilde{\gamma}_0^* - 1)$$

and hence for γ_{N-1}^*, according to formula (4.1). Lemma 6 is thus proved by induction.

As in Lemma 3 we see that Lemma 6 implies the following:

(1) If $f_k > 0$ and $0 < g_k < 1$ for all natural numbers k, then all γ_n^*, $n \leqslant 0$, are positive. (Already known from proof of Lemma 5.)

(2) If $f_k > 0$ and $0 < g_k < 1$ for all natural numbers k, then for all $n \leqslant 0$, γ^* is a strictly increasing function of all f_k's and g_k's in the formula for γ_n^*. On the other hand, for fixed absolute values of all f_k and g_k, $|g_k| < 1$, the $|\gamma_n^*|$'s are maximal when $f_k \geqslant 0$ and $g_k \geqslant 0$ for all k. This implies in particular that in $\mathscr{T}_{R,C}$ we generally have for all $n \leqslant 0$

$$|\gamma_n^*| \leqslant q_n^*,$$

where the q_n^*'s are defined in (4.2).

(3) For all $n \leqslant -1$ we have

$$\tilde{q}_n^* \leqslant \frac{q_n^*}{R}.$$

From the properties 1 and 3 it follows by using Lemma 4 that

$$q_{n-1}^* \leqslant \frac{2CR}{(R^2 - C)(R - C)} q_n^* + \frac{1}{R - C} \sum_{k=n+1}^{0} q_{n-k}^* q_k^*, \quad n \leqslant 0. \tag{4.3}$$

Let $\langle t_n^* \rangle_{n=0}^{-\infty}$ be a sequence of positive numbers, defined by

$$t_0^* = \frac{R + C}{R - C}, \quad t_{n-1}^* = \frac{2CR}{(R^2 - C)(R - C)} t_n^* + \frac{1}{R - C} \sum_{k=n+1}^{0} t_{n-k}^* t_k^*, \quad n \leqslant 0.$$

Then it follows from property 2 and (4.3) that

$$|\gamma_n^*| \leqslant t_n^* \qquad \text{for all} \quad n \leqslant 0. \tag{4.4}$$

We shall now study the formal series

$$T^*(z) = \sum_{-\infty}^{n=0} t_n^* z^n. \tag{4.5}$$

We first rewrite the recursion formula for t_{n-1}^*:

$$(R - C)\, t_{n-1}^* = K t_n^* + \sum_{k=n}^{0} t_{n-k}^* t_k^*, \qquad n \leqslant 0,$$

where

$$K = \frac{2RC}{R^2 - C} - \frac{R + C}{R - C}.$$

For $T^*(z)$ we thus have the formal identity

$$(R - C)\left(T^*(z) - \frac{R + C}{R - C}\right) = Kz^{-1}T^*(z) + z^{-1}T^*(z)^2,$$

or in rewritten form

$$T^*(z)^2 - [(R - C)z - K]\, T^*(z) + (R + C)z = 0.$$

This shows that $T^*(z)$ represents the holomorphic function

$$T^*(z) = \tfrac{1}{2}[(R - C)z - K - ([(R - C)z - K]^2 - 4(R + C)z)^{1/2}] \tag{4.6}$$

in a neighborhood of ∞. (The choice of $-$ follows from $\lim_{z\to\infty} T^*(z) = (R + C)/(R - C)$.)

We shall find the singularity z_0^* of largest absolute value. z_0^* is the root of largest absolute value in the equation

$$[(R - C)z - K]^2 - 4(R + C)z = 0,$$

i.e.,

$$z_0^* = \frac{1}{(R - C)^2}[K(R - C) + 2(R + C) + 2(K(R^2 - C^2) + (R + C)^2)^{1/2}]. \tag{4.7}$$

When $C \to 0$, we have that $K \to -1$ and $z_0^* \to 1/R$. Hence, for any $\epsilon > 0$ there is a $C > 0$, such that for any continued fraction in $\mathscr{T}_{R,C}$ the corresponding series at ∞ represents a holomorphic function in $|z| > 1/R + \epsilon$. This completes the proof of Theorem 1.

As in Section 3 we shall make two observations for later use.

Observation 3. For any continued fraction in $\mathscr{T}_{R,C}$ the corresponding series at ∞ represents a holomorphic function in some neighborhood of ∞. (This is easily seen from (4.6) and (4.7)). But, contrary to the correspondence at 0, there is here no fixed neighborhood of ∞ that will do for all continued fractions in

$$\bigcup_{0 \leqslant C < R} \mathscr{T}_{C,R}\,.)$$

Observation 4. Let $R > 1$ and $\epsilon > 0$ be given. Let furthermore $C \in (0, R)$ be such that $T^*(z)$ is holomorphic in the domain $|z| > 1/R + \epsilon$ (also at ∞). Then the same holds for any $L^*(z)$ corresponding at ∞ to a continued fraction in $\mathscr{T}_{R,C}$. Let $r > 0$ be such that $1/r > 1/R + \epsilon$. Then for any such $L^*(z)$ the following holds in $|z| \geqslant 1/r$:

$$\begin{aligned} |L^*(z) + 1| &= \left| c_0^* + 1 + \sum_{-\infty}^{n=-1} c_n^* z^n \right| \\ &\leqslant |1 - \gamma_0^*| + \sum_{-\infty}^{n=-1} t_n^* r^{-n} \\ &\leqslant \frac{2C}{R - C} + T^*\left(\frac{1}{r}\right) - t_0^* = T^*\left(\frac{1}{r}\right) - 1. \end{aligned}$$

This gives

$$|L^*(z) + 1| \tag{4.8}$$
$$\leqslant \frac{4C/r + 2(1 + K)}{(R - C)/r - K - 2 + (((R - C)/r - K)^2 - 4(R + C)/r)^{1/2}}.$$

5. An Extension of Theorem 1

In the remark after the definition of $\mathscr{T}_{R,C}$ it is pointed out that any continued fraction in $\mathscr{T}_{R,C}$ has a corresponding pair $(L(z), L^*(z))$ of formal Laurent series (1.2). From the observations 1 and 3 we know that $L(z)$ represents a holomorphic function L in some neighborhood of the origin and that $L^*(z)$ represents a holomorphic function L^* in some neighborhood of ∞. We have furthermore that any "tail"

$$\mathop{\mathrm{K}}_{n=N}^{\infty} \frac{F_n z}{1 + G_n z}$$

of a continued fraction in $\mathscr{T}_{R,C}$ is a continued fraction in $\mathscr{T}_{R,C/R^{N-1}}$ and hence has a corresponding pair $(L_N(z), L_N^*(z))$ of formal Laurent series (1.2). To any $\epsilon \in (0, R - 1)$ there is an N, such that $L_N(z)$ represents a holomorphic function L_N in $|z| < R - \epsilon$ and that $L_N^*(z)$ represents a holomorphic function L_N^* in $|z| > 1/R + \epsilon$. Between (L, L^*) and (L_N, L_N^*) we have the obvious relations

$$L(z) = \frac{F_1 z}{1 + G_1 z} + \cdots + \frac{F_{N-1} z}{1 + G_{N-1} z + L_N(z)}$$

and the same with *. Hence L has a meromorphic extension to $|z| < R - \epsilon$ and L^* a meromorphic extension to $|z| > 1/R + \epsilon$. Since $\epsilon > 0$ can be taken arbitrarily small, we have the following result:

THEOREM 2. *To any continued fraction in* $\mathscr{T}_{R,C}$ *the corresponding pair* $(L(z), L^*(z))$ *of formal Laurent series represents a pair of functions* (L, L^*), *where* L *is holomorphic in some neighborhood of* 0 *and has a meromorphic extension to* $|z| < R$, *and where* L^* *is holomorphic in some neighborhood of* ∞ *and has a meromorphic extension to* $|z| > 1/R$.

6. CONVERGENCE

We shall need a result from [7], for our purposes rephrased as follows:

Given $c_1 \neq 0$, and let r and ρ be any two positive numbers such that

$$r > \frac{2}{|c_1|}, \qquad \rho < \frac{1}{2|c_1|}. \tag{6.1}$$

Then there exist numbers $\alpha > 0$, $\beta > 0$, such that if $(L(z), L^*(z))$ is a pair (1.2) of formal Laurent series with that particular c_1-value, then the following holds:

If $L(z)$ represents a holomorphic function L in $|z| < r$ and

$$|L(z) - c_1 z| \leqslant \alpha \qquad \text{in} \qquad |z| < r, \tag{6.2}$$

and $L^*(z)$ represents a holomorphic function L^* in $|z| > \rho$ and

$$|L^*(z) + 1| \leqslant \beta \qquad \text{in} \qquad |z| > \rho, \tag{6.3}$$

then a corresponding general T-fraction exists, is limit-periodic with $F_n \to F \neq 0$ and $G_n \to -F$ as $n \to \infty$, and converges to $L(z)$ locally uniformly in $|z| < 1/|F|$ and to $L^*(z)$ locally uniformly in $|z| > 1/|F|$.

(In [7] is used a slightly different normalization on L, i.e., $L(z) = 1 + c_1 z + \cdots$, and hence the boundedness conditions in [7] look differently: $|L(z) - 1 - c_1 z| = \alpha$, $|L^*(z)| = \beta$.)

Let $R > 2$. Then for all sufficiently small $C \in (0, R)$ we also have

$$R > \frac{2}{1 - C/R},$$

and there is an r, such that

$$R > r > \frac{2}{1 - C/R}. \tag{6.4}$$

Furthermore, since $1/R < \frac{1}{2}$, we have for all sufficiently small $C \in (0, R)$ that

$$\frac{1}{R} < \frac{1}{2(1 + C/R)},$$

and there is an r, such that

$$\frac{1}{R} < \frac{1}{r} < \frac{1}{2(1 + C/R)}. \tag{6.5}$$

We shall assume that $C \in (0, R)$ is small enough to ensure $R > 2/(1 - C/R)$ and $1/R < 1/[2(1 + C/R)]$ at the same time, and that r is picked to satisfy (6.4) and (6.5) simultaneously. Then obviously (6.4) and (6.5) hold for any smaller value of C. Since for all continued fractions in $\mathscr{T}_{R,C}$ the first coefficient c_1 of $L(z)$ must satisfy the inequality

$$1 - \frac{C}{R} \leqslant |c_1| \leqslant 1 + \frac{C}{R},$$

it follows that for any continued fraction in $\mathscr{T}_{R,C}$ the inequalities (6.1) hold with $\rho = 1/r$.

When $C \to 0$, the right-hand side of (3.15) and of (4.8) both tend to zero (since $C \to 0 \Rightarrow K \to -1$), and hence (6.2) and (6.3) are satisfied for all sufficiently small C. Hence we have the following theorem on convergence (since here $F = 1$):

THEOREM 3. *Let $R > 2$. Then for all sufficiently small C any continued fraction in $\mathscr{T}_{R,C}$ will converge locally uniformly in $|z| < 1$ to $L(z)$ and locally uniformly in $|z| > 1$ to $L^*(z)$.*

Remarks. An extreme example is the general T-fraction

$$\frac{z}{1 - z} + \frac{z}{1 - z} + \frac{z}{1 - z} + \cdots. \tag{6.6}$$

This continued fraction is in any $\mathscr{T}_{R,C}$. It is easy to see that (6.6) corresponds to

$$(L(z), L^*(z)) = (z, -1),$$

and that it converges to z locally uniformly in $|z| < 1$ and to -1 locally uniformly in $|z| > 1$. (The convergence properties as well as the correspondence properties are most easily established by calculating the explicit expressions for the approximants.)

It is likely that the conclusion of Theorem 3 also holds for R-values less that 2, but the method of [7] cannot provide any better value. Based upon the results for ordinary T-fractions [1] there is reason to believe that the conclusion holds for all $R > 1$.

In the paper [7] is briefly mentioned the possibility of accelerating the convergence and increasing the domain of convergence in the limit-periodic case by "modifying" the approximants. In the present case this means to study the sequence $\langle S_n(z)\rangle$ in a neighborhood of the origin, and $\langle S_n(-1)\rangle$ in a neighborhood of ∞, where

$$S_n(w) = \frac{F_1 z}{1 + G_1 z} + \frac{F_2 z}{1 + G_2 z} + \cdots + \frac{F_n z}{1 + G_n z + w}.$$

($\langle S_n(0)\rangle$ is the sequence of ordinary continued fraction approximants.) In an unpublished paper [6] on ordinary T-fractions it is proved (under similar conditions as in the present paper), that the "0-modification" (i.e., replacing $S_n(0)$ by $S_n(z)$) extends the convergence to $L(z)$ from the unit disk $|z| < 1$ to the disk $|z| < R$ minus the poles, and uniformly on compact subsets of that domain. It seems likely that this also will be the case for general T-fractions in $\mathscr{T}_{C,R}$ and also that the "∞-modification" will provide convergence to $L^*(z)$ in $|z| > 1/R$ minus poles. Those questions and also estimates of convergence acceleration will be discussed in a subsequent paper.

7. Final Remarks

As stated in the introduction the present paper originates from properties of general T-fractions, corresponding to a pair $(L(z), L^*(z))$ of Laurent series satisfying certain boundedness conditions. This restricts the discussion to general T-fractions, satisfying conditions of type (1.3). A discussion of limit-periodic general T-fractions in general (where $F_k \to F$, $G_k \to G$) is beyond the scope of the present paper. By using [8] in the discussion of correspondence and [5, p. 93] in the discussion of convergence it is not hard, however, to prove that such a T-fraction, under rather mild conditions on F_k and G_k, will correspond to a pair $(L(z), L^*(z))$ and converge to $L(z)$ and $L^*(z)$ in neighborhoods of 0 and ∞.

For the theorems in the present paper, however, the property $G = -F \neq 0$ is essential, and so is the rate at which $F_k \to F$ and $G_k \to -F$.

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